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This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematics.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

An S. P. E. E. Project

In an article which appeared in the S. P. E. E. Journal in 1933, President Doherty of Carnegie Tech gave as the general objectives of engineering:

"An undergraduate course in engineering should train students to become engineers and not train engineers; it should develop a foundation for later professional specialization, yet it should constitute, at the same time, a broad balanced education. It should be a bilateral program involving, on the one side, the mastery of the fundamental principles of the mathematical and physical sciences and a *rigorous discipline in reasoning in which these principles are applied to the analysis of problems*;.....". (Italics are mine).

In the past, mathematics teachers have been satisfied to teach the fundamental principles of mathematics and leave it to the engineering teachers to develop the ability to apply these principles to the analysis of problems instead of making a beginning in this development during the first two years of engineering. This attitude has been paralleled with the student's study of his several courses by *compartments*. The result has been that students on reaching the junior level often are unable to apply mathematical principles unless the teacher assists them in analyzing the problem and tells them what mathematical (and physical) principles are involved.

The fact that the mathematical problems assigned as drill have been essentially one-step or single-idea problems has contributed to this delinquency. Also, the mathematical problems have often been of a geometric character and have not always assisted the student to develop the ability to analyze and apply fundamental engineering principles.

Thus, there is need of coordinating the teaching of mathematics, physics, and the other basic engineering courses; not only to break down the compartment study habit, but also to yield a better general understanding and attitude.

Last summer the mathematics section of the Society for the Promotion of Engineering Education voted to form a national committee to collect engineering applications of elementary mathematics.

This committee has been selected and consists of Professors Cell (N. C. State) Chairman, Hazeltine (Stevens Institute), Krathwohl (Armour Institute), Holl (Iowa State), Churchill (Michigan University), Teare (Yale), Weaver (Ohio State), and Burrington (Case).

This committee is seeking problems of three types:

1. Problems suitable for direct assignment to freshman, sophomore, and junior engineering students in their mathematics courses. These problems should make use of fundamental engineering principles of a type familiar to the student at the appropriate stage of his development. For example:

(*a*) A brick-shaped weight (96 lbs.) rests on a horizontal plane and is to be pulled by a horizontal force of 100 lbs. Friction (opposing motion) is a linear function of the time. The coefficient of friction when $t=0$ sec. (moment at which the 100 lb. force is applied) is $\frac{1}{2}$ and 5 seconds later it is $\frac{1}{4}$. (1) Derive a formula for the acceleration at any time. For what range of values of the time does this problem make sense if considered as a physical problem? (2) Assuming that the velocity and distance are both zero when $t=0$, obtain the equation of rectilinear motion and sketch for the range of values of time previously given. (To be given with constant of integration problems in calculus).

(*b*) Given a bridge truss (anyone would do). Find the distances from some of the pins to various members of the truss. (To be given in analytic geometry in connection with the distance from a line to a point).

2. Problems suitable for classroom use—the teacher to direct the analysis of the problem—the class to assist in the solution as directed by the teacher. For example:

A weight (128 lbs.) is suspended from the ceiling of a room by means of a spring (spring constant 400 lbs./ft.). It is pulled downward 1 inch and then released. Neglecting damping, determine the rectilinear equation of motion and show that the motion is simple harmonic motion. Also determine the period and amplitude of vibration. (Suitable for use in calculus after the class has studied the derivatives of $\sin x$ and $\cos x$).

3. Problems that would be of interest to a teacher of engineering sections in mathematics, that could be used for background but which are beyond the ability of the students at that stage of their development or whose solution is too long for educational value.

That such problems will motivate mathematical study is obvious. It is also felt that they will do much toward removing the previous criticisms.

Teachers of engineering mathematics accumulate such problems slowly. Teachers with little engineering background find them difficult to obtain. Hence, the work of this committee is directed to supplying

this need. We hope that within a year a sufficient number of usuable problems will be in the collection to warrant publishing them in manual form.

The committee will welcome assistance from engineering teachers, whether of mathematics or of other courses. Send any problems you think will be of interest, to the chairman of the committee.

North Carolina State College. JOHN W. CELL.

The annual joint meetings of the Louisiana-Mississippi section of M. A. of A. and the Louisiana-Mississippi branch of the National Council of Teachers of Mathematics will be held in Baton Rouge, March 3-4.

Dr. J. F. Thompson of Tulane University, Chairman of the section, and Professor Herbert C. Ervin of Long Beach, Mississippi, Branch Chairman, will preside at the meetings.

Professor E. T. Browne of the mathematics department of University of North Carolina will be the principal speaker Friday night and Saturday morning. A number of mathematical papers will be presented.

Notes on Two Curves Associated with a Variable Triangle

By M. L. VEST
Davis and Elkins College

It is the purpose of this paper to derive the equations of, and briefly discuss, two algebraic curves associated with a variable triangle.

I.

Consider the triangle ABC, and the perpendiculars from each vertex to the side opposite. Designate the feet of the perpendiculars upon the sides BC, CA and AB as A', B' and C', respectively. $A'B'C'$ is the pedal triangle* of the triangle ABC. We desire the locus of a vertex of ABC such that the area of $A'B'C'$ shall be constant.

Let us place the triangle so that B is at the origin of coordinates and BC lies along the $x-$axis. The vertex A is to be variable. The coordinates of the vertices may then be written as $A : (X, Y)$, B: $(0,0)$, $C : (a,0)$.

The perpendicular from the point C has the equation

$$X(x-a)+Yy=0 \tag{1}$$

and the side AB has the equation

$$Yx=Xy. \tag{2}$$

From (1) and (2) we find the coordinates of C' to be

$$C' : \left[\frac{aX^2}{X^2+Y^2} \quad \frac{aXY}{X^2+Y} \right]$$

and in a similar manner we find

$$A' : [X, 0]$$

$$B' : \left[\frac{aY^2}{(X-a)^2+Y^2} \quad \frac{-aY(X-a)}{(X-a)^2+Y^2} \right]$$

*Durell: *Modern Geometry*, page 20.

If we designate the area of triangle $A'B'C'$ by am, then

$$\begin{vmatrix} X & 0 & 1 \\ \dfrac{aY^2}{(X-a)^2+Y^2} & \dfrac{-aY(X-a)}{(X-a)^2+Y^2} & 1 \\ \dfrac{aX^2}{X^2+Y^2} & \dfrac{aXY}{X^2+Y^2} & 1 \end{vmatrix} = 2am.$$

Dropping capitals this reduces to

$$(3) \quad xy(x^3-2ax^2+a^2x+xy^2-ay^2) + m(x^4-2ax^3+a^2x^2+2x^2y^2-2axy^2+a^2y^2+y^4)=0$$

Equation (3) represents the desired locus which is seen to be a quintic curve.

The curve is found to pass through the two fixed vertices B and C, and, since the curve is the locus of the third vertex A, the quintic passes through all the vertices of the variable triangle ABC regardless of its position. The variable vertex A will, in general, lie at an ordinary (non-singular) point of the curve, but there are double points at the fixed vertices B and C. The tangents to the curve at the double points at B : $(0,0)$ and C : $(a,0)$ are

$$x \pm iy = 0 \quad \text{and} \quad x \pm iy = a$$

respectively. The tangents being imaginary, the double points are isolated points. Moreover, since the tangents pass through the circular points at infinity, the double points may be regarded as point circles.

The quintic has three asymptotes, one real and two imaginary, whose equations are

$$y = -m, \qquad x \pm iy = \frac{a}{2}.$$

The real asymptote is an inflectional tangent to the curve at the point (1,0,0). The imaginary asymptotes are tangent to the curve at $(1, \pm i, 0)$, the circular points. Moreover, the curve has a point of inflection at each of these points for which the asymptotes are the inflectional tangents. Hence, the three asymptotes are all inflectional tangents, and the curve has at least three points of inflection on the

line at infinity. The two imaginary asymptotes have a real point of intersection at

$$\left(\frac{a}{2} ,0 \right),$$

the midpoint of the fixed side of triangle ABC.

The line $x=0$ leads to contact at infinity, namely, at the point (0,1,0) where the curve has an ordinary point with the line at infinity as the tangent.

If we put

$$m=\frac{y}{2},$$

the area of the pedal triangle $A'B'C'$ will be equal to that of the original triangle ABC. The quintic thus becomes

$$y(3x^4-6ax^3+3a^2x^2+4x^2y^2+4axy^2+a^2y^2+y^4)=0,$$

degenerating into the fixed side BC of the triangle and a quartic. The quartic passes through the three vertices of ABC. The fixed vertices B and C are double points of the quartic, and are isolated points, the tangents at these points having the equations

$$y=\pm i\sqrt{3}x \quad \text{and} \quad y=\pm i\sqrt{3}(x-a)$$

respectively.

The quartic has four imaginary asymptotes whose equations are

$$x\pm iy=\frac{a}{2} \quad \text{and} \quad \sqrt{3}x\pm iy=\frac{\sqrt{3}}{2}a.$$

The first two are tangent to the curve at the circular points, and the four intersect at

$$\left(\frac{a}{2} , 0 \right),$$

the midpoint of the fixed side of ABC. Also, each of the second pair of asymptotes is parallel to one of the tangents to the curve at each of the double points B and C, i. e., the two asymptotes and four tangents are parallel by threes.

If we consider m as a parameter, the quintic is found to degenerate into comparatively simple forms when $m=0$ and when $m=-y$.

When $m=0$, the area of the pedal triangle then being zero, the equation of the curve becomes

$$xy(x-a)(x^2-ax+y^2)=0,$$

the quintic breaking down into the $y-$axis, the fixed side of the original triangle, the line $x=a$, and a conic. The conic is a circle with its center at the mid-point of the fixed side BC and passing through the two fixed vertices. Since the variable vertex A lies upon the curve, the circle is the circumcircle of ABC when angle A is a right angle. We may then remark that the fixed side of the triangle and the circle erected upon it as a diameter are components of the quintic when the area of the pedal triangle is zero.

When $m=-y$ the equation of the quintic becomes

$$y^3(y^2-ax+a^2)=0.$$

Thus the curve has as its components the line $y=0$ (the fixed side of the triangle) as a triple line, and a parabola having its vertex at the fixed vertex C of the triangle, the fixed side BC as its axis, and the perpendicular to the fixed side at its mid-point as directrix.

II.

Again consider the triangle ABC, the perpendicular bisectors of each side, and the intersection of each perpendicular bisector with the next consecutive side as we move about the triangle in a counter-clockwise direction. Designate the intersections of the perpendicular bisectors of AB, BC and CA with BC, CA and AB as A', B' and C', respectively. We desire the locus of a vertex of triangle ABC such that the area of $A'B'C'$ shall be constant. Let the coordinates be taken as in I.

The perpendicular bisector of side AB then has the equation

$$2Xx+2Yy=X^2+Y^2 \tag{4}$$

and the side BC of the triangle has the equation

$$y=0. \tag{5}$$

From (4) and (5) the coordinates of A' are found to be

$$A':\left[\frac{X^2+Y^2}{2X},\quad 0\right]$$

and in like manner we obtain

$$B' : \left[\frac{a}{2}, \quad \frac{-aY}{2(X-a)} \right]$$

$$C' : \left[\frac{X(X^2+Y^2-a^2)}{2(X^2+Y^2-aX)} \quad \frac{Y(X^2+Y^2-a^2)}{2(X^2+Y^2-aX)} \right]$$

If the area of the triangle $A'B'C'$ be taken as $\frac{m}{8}$ we have

$$\begin{vmatrix} \frac{a}{2} & \frac{-aY}{2(X-a)} & 1 \\ \frac{X(X^2+Y^2-a^2)}{2(X^2+Y^2-aX)} & \frac{Y(X^2+Y^2-a^2)}{2(X^2+Y^2-aX)} & 1 \\ \frac{X^2+Y^2}{2X} & 0 & 1 \end{vmatrix} = \frac{m}{4}.$$

Expanding and dropping capitals we have as the equation of the desired locus.

$$(6) \quad y(x^5-2ax^4-a^2x^3+2x^3y^2+3a^3x^2-2ax^2y^2-a^4x-a^2xy^2 \\ +xy^4+a^3y^2)+m(x^4+x^2y^2-2ax^3-axy^2+a^2x^2)=0$$

which is seen to be a sextic curve.

The two fixed vertices B and C, as well as the generic vertex A, are points of the sextic. However, while A will, in general, be a non-singular point of the curve, the curve is found to have double points at B and C. The tangents to the curve at B have as equations.

$$x=0 \quad \text{and} \quad mx=a^2y.$$

These being real and distinct the double point is a crunode. The tangents to the curve at the second fixed vertex C have as their equations

$$x-a=0 \quad \text{and} \quad m(x-a)=a^2y.$$

These are also real and distinct. However, the first of these tangents has three-point contact with the curve at C, hence the double point is a flecnode.

The four tangents to the double points at the fixed vertices form a parallelogram whose area is m, or eight times the area of triangle $A'B'C'$.

The sextic has six asymptotes, two real and four imaginary. The real asymptotes are the axes

$$x=0 \quad \text{and} \quad y=0.$$

The first is tangent to the curve at the point (0,1,0) where the curve is found to have a point of inflection, the asymptote being the inflectional tangent. The second is tangent to the curve at (1,0,0) where the curve has an ordinary point.

The four imaginary asymptotes have as equations

$$x+iy=0, \quad x+iy-a=0, \quad x-iy=0, \quad x-iy-a=0.$$

These are tangent to the curve by pairs at the circular points, at which the curve has double points, the first two being tangent at $(1,i,0)$, the second pair at $(1,-i,0)$. The four imaginary asymptotes form a configuration $(6_2,4_3)$, a complete quadrangle, having the circular points $(1,\pm i,0)$, the imaginary points

$$\left(\frac{a}{2},\pm i\frac{a}{2}\right),$$

and the fixed vertices B and C of the triangle as intersections. As noted, the curve passes through four of these intersections with a double point at each of the four. We may take the four intersections not at infinity as the vertices of the 4-point. These four points may be arranged in four groups of three each

$$(0,0), \quad (a,0), \quad \left(\frac{a}{2}, \quad i\frac{a}{2}\right)$$

$$(0,0), \quad (a,0), \quad \left(\frac{a}{2},-i\frac{a}{2}\right)$$

$$(0,0), \quad \left(\frac{a}{2}, \quad i\frac{a}{2}\right), \quad \left(\frac{a}{2},-i\frac{a}{2}\right)$$

$$(a,0), \quad \left(\frac{a}{2}, \quad i\frac{a}{2}\right), \quad \left(\frac{a}{2},-i\frac{a}{2}\right).$$

Considering each triplet of points as the vertices of a triangle it is easily shown that the four triangles thus formed have equal areas, the area in each case being

$$\frac{ia^2}{4}.$$

If we put $m=4ay$ the areas of the triangles ABC and $A'B'C'$ will be equal. The equation of the sextic then becomes

$$y(x^5-2ax^4-9a^2x^3+2x^3y^2+7a^3x^2+2ax^2y^2-a^4x-5a^2xy^2+xy^4+a^3y^2)=0$$

showing that the sextic has degenerated into the fixed side BC of the triangle and a quintic. The quintic passes through the vertices of the triangle ABC, having ordinary points at the fixed vertices B and C. The tangents at these points are

$$x=0 \quad \text{and} \quad x=a$$

respectively. It is found that the second of these tangents has four-point contact with the curve at C. That point is, therefore, a point of undulation. There is one real asymptote which is the $y-$axis, and four imaginary asymptotes, these being the tangents to the curve at the double points occuring at the circular points which were discussed for the general form of the curve.

The use of m as a parameter does not seem to produce any particularly simplified forms of the sextic. One case may be mentioned. When $m=-xy$ the equation of the sextic becomes

$$y(2a^2x^3-x^3y^2-3a^3x^2+ax^2y^2+a^4x+a^2xy^2-xy^4-a^3y^2)=0$$

again showing degeneration into the fixed side and a quintic. The quintic again passes through the three vertices of ABC and also through the midpoint of side BC. The curve has an ordinary point at each of the intersections with the fixed side, the tangents at these points all being perpendicular to the fixed side. The quintic has three real asymptotes

$$x=0 \quad \text{and} \quad y=\pm 2a$$

and two imaginary asymptotes

$$2x\pm 2iy=a.$$

The two real asymptotes which are parallel to the $x-$axis are the tangents to the curve at a double point which occurs at (1,0,0) and the imaginary asymptotes are tangent to the curve at the circular points which are ordinary points of the quintic. The latter asymptotes are concurrent with the real asymptote $x=0$ at the vertex B.

Returning now to the derivation of the general curve we note that the vertices of the triangle $A'B'C'$ were gotten by taking the intersection of the perpendicular bisector of each side with the next consecutive side as we moved about the triangle in a counter-clockwise direction. If, instead, we move in the opposite direction, designating

the intersections of the perpendicular bisectors of sides AB, BC and CA with CA, AB and BC as B', C' and A' respectively, and again indicate the area of $A'B'C'$ by $m/8$, the locus of the variable vertex A has the equation

$$(7)\quad y(x^5-3ax^4+2x^3y^2+a^2x^3-4ax^2y^2+2a^3x^2+a^2xy^2 \\ -a^4x+xy^4-ay^4)-m(x^4-2ax^3+x^2y^2+a^2x^2-axy^2)=0.$$

If we apply the transformation

$$T\begin{cases} x=a-x \\ y=y \end{cases}$$

a combination of a translation and a symmetry transformation, to the curve (7) the result will be curve (6). By application of this transformation the singularities of curve (6) which occur at the fixed vertices are interchanged for curve (7), the flecnode being at B and the crunode at C. The other notable points of the curve are translated in like manner.

NOTE ON THE HARMONIC SERIES

$$\sum_{r=1}^{n}\frac{1}{r}=\frac{1}{n}\sum_{r=1}^{n}\frac{C_r^n}{C_{r-1}^{n-1}}$$

If $n=2m$,

$$\sum_{r=1}^{n}\frac{1}{r}=\frac{1}{2m}\sum_{r=1}^{m}\frac{C_r^{2m+1}}{C_{r-1}^{2m-1}}$$

and if $n=2m+1$,

$$\sum_{r=1}^{n}\frac{1}{r}=\frac{1}{2m+1}\left[\frac{C_{m+1}^{2m+1}}{C_m^{2m}}+\sum_{r=1}^{m}\frac{C_r^{2m+2}}{C_{r-1}^{2m}}\right]$$

ISIDOR F. SHAPIRO,
New York City.

The Theory of Roulettes and Glissettes

By Gordon Walker
Cornell University

I. INTRODUCTION

The class of all glissettes includes such locus problems as the involute and evolute. In this paper will be developed a general theory of glissettes, and the relation between glissettes and roulettes.*

DEFINITION: A glissette is defined to be the locus of a point P such that the point P is moving along a curve in a given frame of reference as the frame of reference is allowed to move along a given curve. A glissette may also be defined to be the locus of (*a*) a fixed point, or (*b*) a variable point, belonging to a line of which a fixed point is allowed to slide along a given curve.†

II. THE GENERAL THEORY OF GLISSETTES

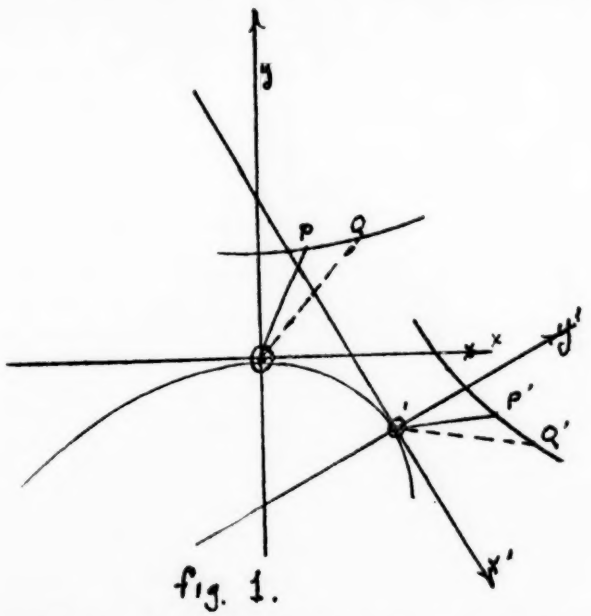


fig. 1.

*For a general theory of roulettes see Gordon Walker, *The General Theory of Roulettes*, National Mathematics Magazine, Vol. XII, p. 21.

†The term *envelope-glissette* is applied to the envelope of a perpendicular line through the variable point.

Consider the line segment $\overline{OP}$ defined by $P=(u(0),v(0))$ on

$$(1)\qquad \begin{cases} x=u(s) \\ y=v(s). \end{cases}$$

Now allow $O=(0,0)$ to move along

$$(2)\qquad \begin{cases} x=f(s) & f(0)=F(0)=F'(0)=0 \\ y=F(s) & \{f'(s)\}^2+\{F'(s)\}^2=1 \end{cases}$$

so that when O is the point $O'=(f(s),F(s))$ P would be defined above as the point $Q=(u(s),v(s))$. Now the line segment $\overline{OP}$ has become the line segment $\overline{O'Q'}$ and the point P has become the point Q'. We say that the locus of P as the point O, of the line segment $\overline{OP}$, moves along (2) is the *glissette* of the point P. To obtain parametric equations of the glissette of P consider the Euclidean transformation:*

$$(3)\qquad \begin{cases} x=h+a\bar{x}-b\bar{y} \\ y=k+b\bar{x}+a\bar{y} \end{cases} \qquad (a^2+b^2=1)$$

which carries the point O into the point O' and carries the $x-$axis (i. e., $x=t$, $y=0$) into the tangent line at O':

$$\begin{cases} x=f(s)+tf'(s) \\ y=F(s)+tF'(s) \end{cases}$$

or specifically the point $(1,0)$ is carried into the point $(f+f', F+F')$ so that

$$0=h+af(s)-bF(s)$$

$$0=k+bf(s)+aF(s)$$

$$1=h+a[f(s)+f'(s)]-b[F(s)+F'(s)]$$

$$0=k+b[f(s)+f'(s)]+a[F(s)+F'(s)]$$

*It would have been possible here to adopt the more restricted form of the transformation where $a=\sin\Theta$, $b=\cos\Theta$. But in this case it is necessary to impose the following restrictions upon the angle Θ:

$\Theta=\alpha+2\sigma\pi n$ where,

(i) α is the positive angle $(<2\pi)$ measured in a counter-clockwise direction from the $x-$axis to the $x'-$axis.

(ii) if $s>0$, $\sigma=+1$
$s<0$, $\sigma=-1$.

(iii) n is the number of complete loops the curve (2) makes between the points O and O'.

and solving for h, k, a, b

$$(4)\qquad \begin{cases} h = -af(s) + bF(s) \\ k = -bf(s) - aF(s) \\ a = f'(s) \\ b = -F'(s). \end{cases}$$

Now, the inverse of the transformation (3) is

$$\begin{cases} \bar{x} = a(x-h) + b(y-k) \\ \bar{y} = -b(x-h) + a(y-k) \end{cases}$$

and putting in the values of a, b, h, k, from (4)

$$\begin{cases} \bar{x} = f(s) + xf'(s) - yF'(s) \\ \bar{y} = F(s) + xF'(s) + yf'(s) \end{cases}$$

when $(x,y) = Q = (u(s), v(s))$ then $Q' = (\bar{x}, \bar{y})$; hence the glissette of P will be given by

$$(5)\qquad \begin{cases} x = f(s) + f'(s)u(s) - F'(s)v(s) \\ y = F(s) + F'(s)u(s) + f'(s)v(s). \end{cases}$$

Example: Consider a circle of radius r, through O, with center $(0, -r)$

$$\begin{cases} x = f(s) = r \sin\dfrac{s}{r} \\ \\ y = F(s) = r\left(\cos\dfrac{s}{r} - 1\right) \end{cases}$$

and the segment of the line $\overline{PQ}$, tangent to the circle at the point Q. To determine the glissette of P such that the distance $\overline{PQ}$ is equal to the length of arc $\widehat{OQ}$. Hence we will take

$$\begin{cases} u(s) = s \\ v(s) = 0 \end{cases}$$

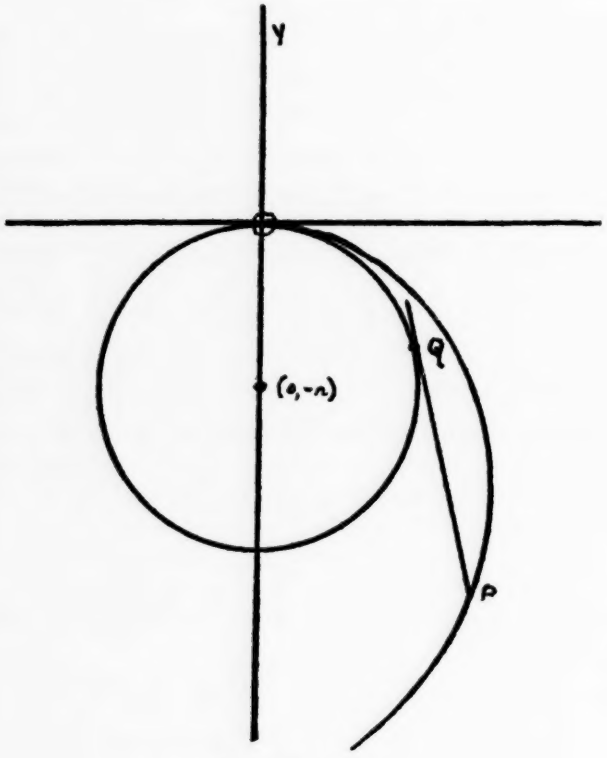


fig. 2

then $f'(s)=\cos\frac{s}{r}$, $F'(s)=-\sin\frac{s}{r}$ and the glissette of P, from (5), will be

$$\begin{cases} x=r\sin\frac{s}{r}+s\cos\frac{s}{r} \\ \\ y=r\left(\cos\frac{s}{r}-1\right)-s\sin\frac{s}{r} \end{cases}$$

which are the parametric equations of the *involute of the circle.**

*The involute of a circle was conceived by Huygens in 1693 when he was considering clocks without pendulums which might be of service on seagoing vessels. In this connection he originated a chronometer in which the involute plays an essential role. (Huygens', *Oeuvres*, Vol. 10, 1905, p. 514). In 1891, for theoretical reasons, the dome of the Royal Observatory at Greenwich was built in the form of a surface generated by the revolution of an arc of the involute of a circle. (*Mo. No. R. Astr. Soc.*, Vol. 51).

Taking $\frac{s}{r}=\alpha$ and the origin of a new coordinate system as $(0,-r)$ we have the familiar form of the equations:

$$\begin{cases} x=r(\sin\alpha+\alpha\cos\alpha) \\ y=r(\cos\alpha-\alpha\sin\alpha). \end{cases}$$

It is known that the involute of a circle is also the roulette of the pole of a logarithmic spiral rolling on a concentric circle.* This would suggest that corresponding to every glissette there exists a roulette giving rise to the same locus, and *vice versa*. It is this correspondence that we shall now investigate.

III. RELATIONSHIP BETWEEN ROULETTES AND GLISSETTES

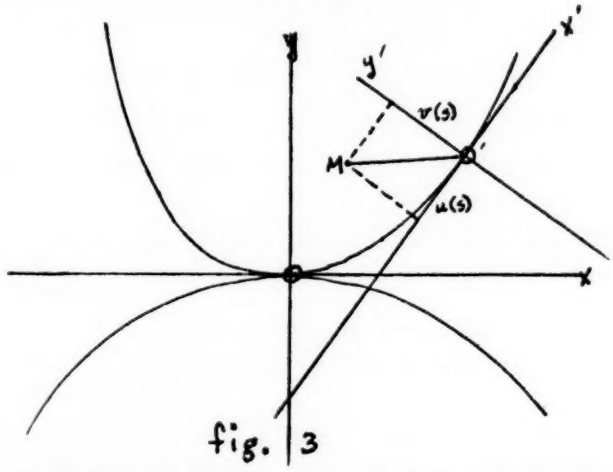


fig. 3

Let the glissette of the point $M=(p,q)$, as the point $O=(0,0)$ of the line segment $\overline{OM}$, moves along the curve

$$(6)\qquad \begin{cases} x=g(s) & g(0)=G(0)=G'(0)=0 \\ y=G(s) & \{g'(s)\}^2+\{G'(s)\}^2=1 \end{cases}$$

be always $M=(p,q)$; to determine $u(s)$ and $v(s)$ we have from (5):

$$\begin{cases} p=g(s)+g'(s)u(s)-G'(s)v(s) \\ q=G(s)+G'(s)u(s)+g'(s)v(s) \end{cases}$$

*Maxwell, *Scientific Papers*, Vol. 1, 1849.

or (7)
$$\begin{cases} u(s)=g'(s)\{p-g(s)\}+G'(s)\{q-G(s)\} \\ v(s)=g'(s)\{q-G(s)\}-G'(s)\{p-g(s)\}. \end{cases}$$

Now consider the line segment $\overline{OP}$ defined by $P=(u(0),v(0))$ on (7). Allow $O=(0,0)$ to move along

(8)
$$\begin{cases} x=f(s) & f(0)=F(0)=F'(0)=0 \\ y=F(s) & \{f'(s)\}^2+\{F'(s)\}^2=1 \end{cases}$$

so that the length of arc $\overset{\frown}{OO'}$, $O=(f(s),F(s))$, is always equal to the length of arc $\overset{\frown}{PQ}$, $Q=(u(s),v(s))$. Then the parametric equations of the glissette of the point P are by (5):

$$\begin{cases} x=f(s)+f'(s)\{g'(s)[p-g(s)] \\ \qquad +G'(s)[q-G(s)]\}-F'(s)\{g'(s)[q-G(s)]-G'(s)[p-g(s)]\} \\ y=F(s)+F'(s)\{g'(s)[p-g(s)] \\ \qquad +G'(s)[q-G(s)]\}+f(s)\{g'(s)[q-G(s)]-G'(s)[p-g(s)]\} \end{cases}$$

or
$$\begin{cases} x=f(s)+\{f'(s)g'(s)+F'(s)G'(s)\}\{p-g(s)\} \\ \qquad +\{f'(s)G'(s)-F'(s)g'(s)\}\{q-G(s)\} \\ y=F(s)-\{f'(s)G'(s)-F'(s)g'(s)\}\{p-g(s)\} \\ \qquad +\{f'(s)g'(s)-F'(s)G'(s)\}\{q-G(s)\}. \end{cases}$$

But, these are the parametric equations of the roulette of the point M as the figure consisting of the point M and the mobile curve (6) is allowed to roll along the stationary curve (8).* Hence we have the

Theorem: The roulette of the point M as the figure consisting of the mobile curve (6) and the point M is rolled along the stationary curve (8) is identical with the glissette of the point P of the line segment $\overline{OP}$, defined by $P=(u(0),v(0))$ on (7), as the point O moves along the stationary curve (8) so that the length of arc $\overset{\frown}{OO'}$, $O'=(f(s),F(s))$, is always equal to the length of arc $\overset{\frown}{PQ}$, $Q=(u(s),v(s))$.

*Gordon Walker, *The General Theory of Roulettes*, National Mathematics Magazine, Vol. XII, p. 24.

To prove the converse theorem consider the glissette of the point P of the line segment $\overline{OP}$ defined by $P=(u(0), v(0))$ on

$$\begin{cases} x=u(s) \\ y=v(s) \end{cases} \tag{9}$$

as $O=(0,0)$ moves along (8) as above. To this end we will take

$$\begin{cases} x=g(s)=s \\ y=G(s)=0 \end{cases} \tag{10}$$

as a mobile curve and $M=(p,q)=(u(s)+s, v(s))$. Then allow the figure consisting of the mobile curve (10) and M to roll along the stationary curve (8) and the roulette of the point M will be

$$\begin{cases} x=f(s)+f'(s)u(s)-F'(s)v(s) \\ y=F(s)+F'(s)u(s)+f'(s)v(s) \end{cases}$$

which by (5) is the glissette of the point P defined by (9), hence

Theorem: The glissette of the point P of the line segment $\overline{OP}$ defined by $P=(u(0), v(0))$ on (9) as $O=(0,0)$ moves along (8) as above is identical with the roulette of the point $M=(u(s)+s, v(s))$ as the figure consisting of the mobile curve (10) and the point M is rolled along the stationary curve (8).

Harmonic Series in Interest Problems

By Isidor F. Shapiro
New York City

The geometric series occurs so preponderantly in compounding of interest, that the case of the harmonic series is apt to be overlooked.

A very practical case, of common application, arises for concerns which put out their funds at interest only in units of, say, one hundred dollars. In other words, the interest does not become available for itself drawing interest until it has accumulated to one hundred dollars. It must therefore be considered "as if" simple interest while under that unit amount; and compounding enters the problem only in the sense that an accretion of interest amounting to the capital-unit draws interest again on itself. This might be called "Complex Interest", or "Compounded Simple Interest".

Let U_1 represent a capital-Unit (say \$100); and let **p** represent the period occupied by the interest-accumulation till the interest amounts to another U_1. Let $U_2=2U_1$; $U_3=3U_1;\cdots\cdots$; $U_n=nU_1$. Then it is manifest that

U_1 becomes U_2 $(=U_1+1U_1)$ in 1 p;

U_2 next becomes $U_3(=\frac{3}{2}\ U_2=U_2+\frac{1}{2}U_2)$ in $\frac{1}{2}$ p;

U_3 next becomes $U_4(=\frac{4}{3}\ U_3=U_3+\frac{1}{3}U_3)$ in $\frac{1}{3}$ p;

.....

and we next attain U_n.....in $\dfrac{1}{n-1}$ p.

If we represent by y_n the time for attaining to U_n, then it is obvious that

$$y_n=p\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots\cdots+\frac{1}{n-1}\right)=p\cdot H_{n-1}.$$

Editorial Note. Let p denote the number of equal interest periods with rate i which an amount u, will require to reproduce itself at simple interest. The series of amounts at the end of each period are $u(1+i)$; $u_1(1+2i)$;.....; $u_1(1+pi)$. Then from the above considerations $1+pi=2$ or $p=1/i$. This value in $y_n=p(H_n-1)$ gives

$$y_n=\frac{H_n-1}{i}.$$

From the conditions of the problem p will increase as i decreases. Also H increases with u and as $n\to\infty$ H is divergent. As a consequence the formula is not bounded.

C. D. Smith.

A Better Elementary Method for Computing Logs and Antilogs

By A. B. SOBLE
Philadelphia, Pa.

We develop here a method of computing logs and antilogs which is simpler and neater than those methods of Chrystal's *Algebra* which do not involve exponential or logarithmic series. The central idea is probably not new.

Extracting the fifth successive square root of 10 we have

$$10^{\frac{1}{32}} = 1.074,607,828.$$

Now $10^{.00001} = \left(10^{\frac{1}{32}}\right)^{\frac{1}{3125}} = \left(1 + .074,607,828\right)^{\frac{1}{3125}} = 1.000,023,026$

by the binomial expansion. Then the following table is developed by simple multiplication:

N	$10^{.0000N}$	$10^{.000N}$	$10^{.00N}$	$10^{.0N}$	$10^{.N}$
1	1.000,023	1.000,230	1.002,305	1.023,293	1.258,925
2	1.000,046	1.000,461	1.004,616	1.047,129	1.584,893
3	1.000,069	1.000,691	1.006,932	1.071,519	1.995,262
4	1.000,092	1.000,921	1.009,253	1.096,478	2.511,885
5	1.000,115	1.001,152	1.011,579	1.122,018	3.162,276
6	1.000,138	1.001,383	1.013,911	1.148,154	3.981,069
7	1.000,161	1.001,613	1.016,249	1.174,897	5.011,868
8	1.000,184	1.001,844	1.018,591	1.202,264	6.309,567
9	1.000,207	1.002,074	1.020,939	1.230,269	7.943,274

Suppose we wanted to find log 274.83. Obviously this is 2+log 2.7483. We see that $10^{.4}$ is the largest number of our table contained in 2.7483. Thus we divide 2.7483 by $10^{.4}$ and get 1.094,119. Now we see that $10^{.03}$ is the greatest number of our table contained in 1.094,119. Thus we divide 1.094,119 by $10^{.03}$ and get 1.021,092. Now we see that $10^{.009}$ is the largest number of our table contained in 1.021,092. Thus we divide 1.021,092 by $10^{.009}$ and get 1.000,150. Now we see that $10^{.00007}$ is the number of our table closest to 1.000,150. Hence, log 274.83 is 2.43907.

Suppose we wanted to find antilog .38,546. The answer is

$$10^{.3} \times 10^{.08} \times 10^{.005} \times 10^{.0004} \times 10^{.00006}.$$

We find these multipliers in our table.

There remains the question of the accuracy of the table. All calculations were carried to nine decimals, the arithmetic being performed and checked on calculating machines capable of handling numbers of ten digits. The table gives six decimal places, but the values toward the end of the table are not that accurate, the errors increasing rapidly as we approach the end of the table. Therefore since $10^{.5}$ is well toward the end, its error gives a good indication of the accuracy of the table. Comparing our nine decimal value for $10^{.5}$ with the square root of 10 correct to nine decimals, we find that the discrepancy is less than .000,002.

Obviously the more decimals to which the calculations are carried, the more accurate will be the table.

"The history of calculating boys is interesting. Mathieu le Coe (about 1664), a boy of Lorraine, could extract cube roots at sight at the age of eight. Tom Fuller, a Virginia slave of the eighteenth century, although illiterate, gave the number of seconds in 7 years, 17 days, 12 hours after only a minute and a half of thought. . . . Ampère, the physicist, made long calculations with pebbles at the age of four. Gauss, one of the few infant prodigies to become an adult prodigy, corrected his father's pay roll at the age of three."

From a footnote in De Morgan's "A Budget of Paradoxes," Vol. I. The Open Court Publishing Company.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON

How a Lady's Dress Influenced Mathematical History

(*A* TRUE STORY)

By B. M. WALKER
Starkville, Mississippi

A young mathematics instructor in a Southern college was taking graduate work leading to the Ph.D. degree, in the University of Chicago, during the early days of that institution. His duties at the Southern college called for nine months of teaching each year. This delayed the young instructor for a longer period than customary, and he was forced to spend several years at work on his thesis: *On the Resolution of the Higher Singularities of Algebraic Curves Into Ordinary Nodes.*

When the thesis leading to the degree of Doctor of Philosophy finally had been finished and was submitted to the department of mathematics, the distinguished professor of the University, who had been directing the research work,* had made all arrangements to spend the following year in Germany and decided he would take the manuscript along with him for study and a careful investigation before passing final judgment on the merits of the paper.

On the return trip from Europe the professor, duly appreciating the value and importance of this manuscript, the only copy in existence which contained a satisfactory solution of this celebrated theorem, was very careful not to put it in his suit cases, fearing it might get lost en route home. To make it perfectly safe and take no chances, he placed the manuscript, with all the detailed data accompanying it, in his wife's trunk with her fine dresses. The valuable trunk was not trusted to an expressman to be hauled alone from the hotel, so the professor got on the express wagon with the driver and accompanied the trunk, delivering it in person, to the steamship officials to be put

*Oskar Bolza.—ED.

on the *Bremen* of the North German Lloyd Line. He then returned to his hotel to get his wife, came down to the dock in a taxi and went aboard the ship. After the *Bremen* had sailed for New York and had been for several hours out at sea, the wife decided she would like to have a certain fine dress out of her trunk for wear that evening. So she asked her distinguished husband to have the trunk brought up to her state-room that she might have access to it. The professor proceeded to find the bursar and made known his wishes and then the trouble began.

The bursar searched through all the baggage and reported to the professor that he was sorry, but there was no trunk on board ship fitting his description. The professor immediately reported the matter to the Captain of the vessel, who ordered the bursar to go back with the professor and carefully go over every piece of baggage on the ship and *find that trunk*. The examination was most carefully made and a report went back to the Captain: that "the trunk is not on board ship," and the bursar returned to his duties.

Then the professor went into action in dead earnest with the Captain and demanded that the ship turn round and go back and *find that trunk*. To this Captain replied that it was not possible, as they were already several miles out at sea en route to New York and the vessel must continue on its voyage without interruption; besides, what a commotion it would create among the passengers for the big steamship to turn round and steam back to port. He would have to make an official report of the matter to the Head Office. The professor could make out a carefully itemized statement of his losses in the trunk and a check would be issued to him to cover in full all his losses, and the ship would continue on its journey to New York.

But this decision by the Captain was not at all satisfactory to the professor, who became still more excited, and as both men were Germans there was no difficulty in speaking German to each other. So the professor in no uncertain language proceeded to tell the Captain that no valuation in dollars and cents could be placed on this trunk, that it contained a piece of mathematical work, the solution of a celebrated problem in mathematics, which had engaged the attention of the mathematical world for years, and the publication of it would bring great honors to the mathematician who first solved it and gave his solution to the world; that the solution had now been made; that he had gone over it in full detail and found it to be satisfactory and correct; that it had been submitted as a thesis for the doctorate in mathematics at the University of Chicago. It was the only copy in existence and was in that trunk. He in person had accompanied the express-

man in delivering the trunk to the steamship officials before he even brought his wife down to the steamer, and now the trunk was lost and *must be found*. He could take no chances that some other mathematician, who might also have been working on this problem, might rush into print and announce the discovery of the correct solution and thus get the credit and the glory, while the University of Chicago and the young instructor, who had spent years in research on this celebrated theorem, would lose out in the deal. There was nothing else to do but to turn the ship round and steam back to Europe and *find that trunk* without fail.

This earnest and forcible plea was too much and too strong for the Captain to withstand, and orders were immediately given to turn the steamship round and proceed back to port. The vessel turned round and began the trip back to Europe to the great consternation of all on board. But back to port it went in full speed and in due time pulled up at the dock.

Then the Captain had the professor go ashore and on the platform show exactly the spot where the trunk had been placed along with other trunks for loading by the steamship officials. The bursar explained that no baggage had been left on the platform when the steamer sailed several hours ago. The Captain reasoned that since the trunk is not on the vessel now, and if it had been put on, it could not have gotten off, then it must be in the sea.

The Captain then ordered the crew to get the drag-nets and the grappling hooks and proceed to drag and grapple for that trunk. The water was deep, the trunk had been under water and down in the muck for several hours then. After considerable time and much hard work, the drag-net struck something deep under water and down the grappling hooks went to bring up whatever it was. With dirty sea water streaming out, up came the trunk to the delight of the anxious professor, who exclaimed: *That is my trunk, put it on deck.* When safe on deck, the professor opened it up and got out the manuscript, thoroughly watersoaked and all stuck together, being written in ink and in long hand. The mass certainly presented an object of despair. The professor felt sick at heart.

The Captain commented: "I am sorry but it is now evidently ruined; however, what do you want to do with it?" The professor replied: "Take it down to the drying room and have it dried out thoroughly, just as it is, but under no circumstances scorch it or burn it." The Captain gave the orders to be carried out under the supervision of the professor. After having lost a half-day or more, the Captain gave the necessary orders for the steamship to proceed again

to sea, bound on its long journey to New York. After the mass of paper had been thoroughly dried, it was done up in clean paper in a nice looking package for the trip home.

The young instructor having received a wire from the distinguished professor to meet him in Chicago on a certain day, after taking into account the time required for passage on the sea and the trip from New York to Chicago, was promptly on hand in a great state of expectancy, not knowing anything of the happenings, but anxious to get the report on his thesis. He was delighted to receive hearty congratulations from the distinguished professor on the satisfactory solution of the celebrated theorem with all the honor and glory that went with it. But then the latter proceeded to say: "I have some bad news now." He related the close approach to a casualty with the manuscript, in graphic terms. Further he remarked: "You are the only man in the world today, who can take that mass of manuscript and dissect it and rewrite that paper. I would not attempt it myself, and I want you to get to work on it at once, so that we shall take no chances of losing the credit. In about thirty days' time the paper had been rewritten. When it was published congratulations came from twenty-three countries of the world.

That is *how a lady's desire to wear a fine dress saved a valuable contribution to science.*

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

Vexing Minor Problems of the Mathematics Curriculum

By CECIL B. READ
The Municipal University of Wichita, Wichita, Kansas

Many of the problems here presented are unquestionably minor; just as truly are they vexing. The accumulation of many such minor but vexing problems often leads the college teacher to ask whether any solution is possible. In some cases we feel that at the University of Wichita we have at least a partial solution; in other cases we are still at a loss.

A frequently occurring problem is the question of what provision is to be made for those students who, although possessing sufficient credits to enter college, lack certain mathematics courses prerequisite for further mathematical work. For example, we find students entering with no solid geometry yet planning on an engineering career; we find students entering with neither algebra nor geometry but still having ambition to major in some physical science; many students come from small high schools where only two years of mathematics is offered.

With respect to the solid geometry it has been our policy to avoid if possible the offering of this course for credit. Once we offered the course to a small group of engineers but allowed no credit. Our most satisfactory solutions have been either to have the student so plan his schedule that he can take this course at one of the local high schools or to have him take the course by correspondence, preferably from the extension division of some state university.

With respect to the failure to offer algebra or plane geometry the situation is somewhat more involved. Even if the student takes the work by correspondence or from a local high school his later mathe-

(A summary of a paper presented before the Kansas Section of the Mathematical Association of America, April 2, 1938.)

matics work will be so delayed that we really do not have an effective solution. In many cases the student is forced to change his entire plans for his future career. With this situation one indeed wonders if this can be called a minor problem. Certainly high school advisors need a long-time point of view and should be very careful before permitting a student to omit any or all mathematics courses.

For some years we have been bothered with the problems of an insufficient number of members on our teaching staff. We have, of course, resorted to the rather common plan of alternating junior and senior courses. Formerly, with more instructors, we offered a course in analytical geometry for liberal arts students and another course for engineering students. The latter course devoted a portion of the time to elementary solid analytics and carried a greater number of hours of credit. With the necessity for reducing the teaching load we hit upon the idea of offering plane and solid analytic geometry at the same time. One day of each week is devoted to a study of solid geometry. Students who do not wish this extra credit do not attend class on this day. Surprising as it may seem this plan has worked out very well. It makes possible the development of a very interesting parallelism between the formulas of plane and solid geometry. We have a similar situation with respect to calculus courses for liberal arts and engineering students. We have not attempted to combine courses here, but it might be a possibility.

The question of combining two courses offers several interesting possibilities. Our course in college algebra, three semester hours credit, has as a prerequisite one and one-half units of high school algebra. For those students who can present only one unit of high school algebra we offer a course in college algebra for five semester hours credit. The assumption is that the additional two hours cover the material of the third semester in high school. It has been found distinctly advantageous for the student who is weak in a three-hour course to take the five hour course for three hours credit. This raises the question: would it be possible to offer the two courses as one, permitting those students who have the proper prerequisites to attend only three days out of five? This would, of course, require an exceptionally careful selection and arrangement of material, but I do not believe it is impossible.

One of the most annoying situations that can be encountered in a college course is the lack of proper foundation. There may be many basic causes of this. Sometime ago several of our instructors had the privilege of hearing a description of the material covered in certain high school courses in algebra. Greatly to our surprise we found that

the high schools were covering, or shall we say, attempting to cover, a large portion of the material which we were attempting to teach in college algebra. Here is indeed a vexing problem: is the best result obtained by continued repetition of the material, or would it be better to get a thorough knowledge of the fundamentals before proceeding to more advanced topics? Personally, I should much prefer that an entering student be positive of the fact that x^2 times x^5 is x^7 and not x^{10} than that he have some slight acquaintance with mathematical induction or the theory of equations.

As an example of the lack of rigor so frequently encountered, I find it not at all unusual for students to say that since, three divided by zero is infinity and four divided by zero is infinity, it follows that $3/0+4/0=2\infty$. I have been seeking methods of impressing upon the student the proper procedure in this and similar situations. I have, for example, offered to give a high mark in the course to any student who can find for me in any recent text book the *unqualified* statement that a constant divided by zero is infinity. So far, the offer has not been taken up.

Many other minor problems might be mentioned. No doubt all of you have just as vexing problems. For example, how can one be sure that he has two equivalent forms of the same examination? What is the relative value of work prepared outside the class? How large may a section be for effective teaching? I shall conclude by commenting briefly on this last question.

With decreased instructional staff and increasing number of students, sections often consist of thirty-five, forty, fifty or more students. Granted that ordinary teaching methods are ineffective here, what can be done? Since with such a large section the teaching is largely by the lecture method, what would happen if we permitted unlimited enrollment? It might be possible to put a master teacher in charge of such a group, releasing other instructors for purposes of individual help. This might permit the grading of papers by experienced teachers rather than by student assistants. It might make possible a wider use of the plan of conference periods. One day a week might be devoted to small informal discussion groups or quiz sections. Possibly with only one lecture to give rather than a repetition of a discussion of the same material by the same instructor in two or three sections of the course, we might find incentive for more effective presentation. With such a plan an instructor working as an apprentice under a master teacher might obtain far more valuable methods of teaching than are now available in any formal course. It does not necessarily follow that I am favoring such a plan; I merely present some of its possible advantages for your consideration.

A Problem of Presentation in Trigonometry

By Louis C. Karpinski
University of Michigan

The fundamental definitions and formulas of trigonometry are presented to those planning to teach mathematics in the first course in algebra, in a separate course in trigonometry, in analytics, and in the calculus. Despite these variant presentations, many of these students hesitate about sine, cosine, and tangent, on the functions of thirty and forty-five degrees, and particularly on the functions of the related angles. The writer believes that widespread agreement upon the form of presentation of the basic ideas would result in far greater success on the part of the pupils in understanding and reproducing the elementary formulas.

In connection with hundreds of attempts to train people to execute with maximum speed repetitive mechanical operations, certain basic principles have been established which appear to have validity even in the intellectual field. It is accepted as axiomatic that the proper way to acquire speed is to begin immediately with the methods and motions that have proved successful in producing exceptional achievement. In typing, for example, one begins with the proper fingering and other motions even though this slows up the initial production.

In trigonometry the use of the expressions, side opposite, side adjacent, and hypotenuse, instead of x, y, and r (or v), undoubtedly constitutes the major hindrance to the immediate memorizing of the general and the particular relationships.

Most mathematicians would agree that the primary basic definitions and formulas are the following:

$$\sin\theta=\frac{y}{r};\quad \cos\theta=\frac{x}{r};\quad \tan\theta=\frac{\sin\theta}{\cos\theta}\left(=\frac{y}{x}\right);$$

the three reciprocal relationships, with minor stress; $\sin^2\theta+\cos^2\theta=1$ and $1+\tan^2\theta=\sec^2\theta$. With weaker pupils, particularly, the mental effect of reducing the apparent number of formulas to five or six, instead of ten or twelve, plays an important part in inspiring the confidence of the pupil in his ability to master the subject.

There is no good reason to begin the right triangles with the sides a, b, and c, whereas the immediate use of x, y, and r connects naturally with the extension to the general angle. At the same time even the elementary school pupil has immediately the connection with graphs, reinforcing the trigonometric ideas. For the student who goes on with algebra the same ideas are involved in the graphical representation, making that presentation simpler, and strengthening the trigonometric relationships. It is by the interplay between the geometrical and the algebraical ideas that the trigonometric ideas should be fixed so firmly in mind that a larger percentage of our students will be able to recall more quickly and certainly the fundamental ideas.

So far as the angles of 30°, 45° and 60° are concerned, the connection which seems most likely to remain with the alert pupil is that involving the first two regular polygons, the equilateral triangle and the square. The major stress in the presentation should be upon the equilateral triangle, split by the $x-$axis into two equal triangles, having the angles of 30° and $-30°$, or 30° and 330°, in position to read the trigonometric functions. Rotating this triangle about the origin through 180° the completed figure gives 150° and $-150°$ in a position to give the functions; placing the same triangle vertically with its base upon the $x-$axis and with one vertex at 0, the four positions present 60° and the related angles. Similarly the square with its diagonal is an easier connection than the isosceles right triangle to arrive at the functions of 45° and its related angles.

To some teachers the above remarks may seem to border upon the trivial. Anyone who has taught some hundreds of freshmen and sophomores—college or high school—knows that even half-a-dozen presentations are not sufficient to give successful understanding to forty out of fifty of these youngsters. However, most of us have the conviction that some processes of presentation are better than others, leading to more consistent results.

In many of the large universities the number of students taking elementary courses is sufficiently large so that worthwhile experiments in presentation could be devised which would lead to somewhat definite conclusions. The initiation of such experiments would lead to a more realistic investigation of methods of presentation, of the desirability of frequent tests and actual consideration of variant types of tests. Extended over three or four years such tests would undoubtedly lead to a more intelligent discussion of the possibilities of making the mathematical work of the freshman and sophomore far more vital to much larger groups of students.

Mathematical World News

Edited by
L. J. ADAMS

During the first week in December, 1938 Professor R. C. Yates, University of Maryland, visited Louisiana State University at Baton Rouge, Louisiana, and delivered a series of lectures before faculty and student groups on the following subjects: *Trisection*, *The Linkages of Mechanics*, and *The Geometry of Motion*. On November 26, 1938 Professor Yates gave a talk on *Linkages* at the meeting of the Philadelphia section of the Mathematical Association. These were invited addresses.

The American Mathematical Society has appointed a committee on publicity.

The Princeton University Press announces the publication of *The Classical Groups, Their Invariants and Representations* by Hermann Weyl, and *Topological Groups* by Leon Pontrjagin. The latter is a translation from the Russian by Emma Lehmer.

On one of the December programs of the *March of Science*, broadcast over the National Broadcasting System, Professor E. J. Moulton gave an address on the origin and purpose of mathematical societies. Professor Moulton is editor of the American Mathematical Monthly.

The Boletin Matematico is a mathematical journal published in Argentina. It was founded by Bernardo I. Baidaff eleven years ago. It is edited by Professor Baidaff, with the help of an extensive editorial board composed of mathematicians of many different countries. On the cover is a list of the mathematicians who have contributed to the magazine since its origin. There are several mathematicians of the United States included in this list. The editors state that they give preferential attention to brief articles of approximately two hundred lines of seventy letters each. Correspondence should be addressed to Avenida de Mayo 560, Buenos Aires, Republica Argentina.

The second annual William Lowell Putnam mathematical competition will be held on Saturday, March 4, 1939. It is open to under-

graduates in the United States and Canada who have not received a degree. Complete details of the competition can be found in the January, 1939 issue of the American Mathematical Monthly.

The Chauvenet Prize has been awarded to Professor G. T. Whyburn, University of Virginia, for his article *On the Structure of Continua*, in the 1936 volume of the *Bulletin* of the American Mathematical Society. This is an expository article dealing with the theory of the structure of continua, and contains many of the results obtained by Professor Whyburn, who has done an important job in contributing to the extension of this field of topology. The Chauvenet Prize is awarded every three years for the best expository paper published in English by a member of the Mathematical Association. It is a cash award of one hundred dollars. Since the prize was established, five mathematicians have been recipients, namely, G. A. Bliss, T. H. Hildebrandt, G. H. Hardy, Dunham Jackson, and now G. T. Whyburn.

Professor G. C. Evans has been elected president of the American Mathematical Society.

Professor W. M. Whyburn, University of California at Los Angeles, has been appointed section committeeman for the section on mathematics of the American Association for the Advancement of Science, of which Professor Marston Morse, Princeton University, is chairman.

Dr. Fabian Franklin, one time professor of mathematics in the Johns Hopkins University, died on January 8, 1939.

Professor P. D. Edwards, Ball State Teachers College, has been appointed chairman of the division of mathematics in the Indiana Academy of Science for the year 1939.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, College Park, Md.

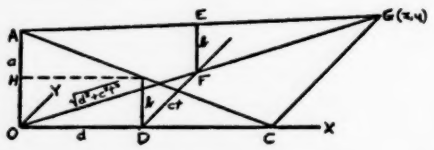
SOLUTIONS

Late Solutions: No. 222 by *Johannes Mahrenholz*, No. 33 by *Fred Marer*.

No. 38. Proposed by *E. M. Shirley*, Louisiana Polytechnic Institute.

An arc light a units high stands in the middle of a street while a man b units high ($b<a$) walks along a sidewalk at c feet per second. State the parametric equations which give the position of the end of the man's shadow at any time t. Along what locus does the end of the shadow move and at what velocity along the locus? Show analytically that the man's velocity must always be less than that of the end of the shadow.

Solution by *C. W. Trigg*, Los Angeles City College.



Let the street by $2d$ feet wide and let the units in which a and b are expressed be in feet. Choose the foot of the arc-light as the origin and the perpendicular from it to the sidewalk as the $X-$axis. Let $t=0$ when $y=0$.

For any location, F, of the man on the sidewalk, the end of the shadow will be at $G(x, y)$, and $OF=\sqrt{d^2+c^2t^2}$. $OG:FG::a:b$, so $OG=a(OG-\sqrt{d^2+c^2t^2})/b$, whence $OG=a\sqrt{d^2+c^2t^2}/(a-b)$.

Then $x : d :: OG : OF :: y : ct$, so $x = ad/(a-b)$ and $y = act/(a-b)$.

Hence the end of the shadow moves along a line parallel to the sidewalk (assuming no walls are in its path) at a rate of $ac/(a-b)$ feet per second. Since $b < a$ this rate is always greater than c feet per second. Furthermore, as $a/(a-b)$ is a pure number when a and b are measured in the same units, the restriction on the nature of the unit may be removed.

If the parametric equations are not required, the coordinate system is not needed, for $AO : EF :: OG : FG$ and $AO : BD :: OC : DC$. Now $EF = BD$, so $OG : FG :: OC : DC$. Hence GC is parallel to FD and $GC : FD :: OC : OD :: OC : HB :: AO : AH :: a : (a-b)$. And since $FD = ct$, $GC = act/(a-b)$, thus leading to the conclusion previously reached.

Also solved by *Walter B. Clarke, Albert Farnell,* and *W. Irwin Thompson.*

No. 218. Proposed by *Walter B. Clarke,* San Jose, California.

Given a circle with two intersecting chords AB and CD. Locate a point on either chord equidistant from the other chord and from the circumference.

Solution by *C. W. Trigg,* Los Angeles City College.

Analysis:

Let the intersection of the chords be M, and the required point, P, be on MD. Let the foot of the perpendicular from P to AB be Q and the extremity of the radius through P be R. Set $CM = b$, $MP = x$, $MD = a$, $OR = r$, and $\angle BMD = \theta$. Then $QP = x \sin \theta = PR$, so

$$x \sin \theta \ (2r - x \sin \theta) = (b+x)(a-x).$$

When this equation is solved for x,

$$x = \frac{-(2r \sin \theta - [a-b]) \pm \sqrt{[2r \sin \theta - (a-b)]^2 + (2a \cos \theta)(2b \cos \theta)}}{2 \cos^2 \theta}.$$

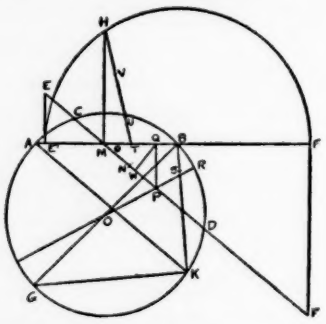
Here, only the positive sign will be employed. To find another point, P', on CM, interchange a and b in the expression for x.

Construction and Proof:

Extend MC to E and MD to F so that $ME = 2b$ and $MF = 2a$. Project EF on to AB, then $E'M = 2b \cos \theta$ and $F'M = 2a \cos \theta$. Erect

a perpendicular to $E'F'$ at M meeting the circle constructed on $E'F'$ as diameter, at H. Then $HM=\sqrt{2a\cos\theta\cdot 2b\cos\theta}$.

Draw a parallel to CD through A, meeting the circle at K. Then $BK=2r\sin\theta$. [Since $AK\|CD$, $\theta=\angle BMD=\angle BAK=\angle BGK$, where BOG is a diameter and BGK is a right triangle].



On MD strike off $MN=MC=b$, and on KB strike off $KS=ND=a-b$. Then $BS=2r\sin\theta-[a-b]=y$. On MB lay off $MT=BS$. Then $TH=\sqrt{y^2+(2a\cos\theta)(2b\cos\theta)}=z$. On TH lay off $TU=y$ and bisect HU at V. Then $HV=\frac{1}{2}(-y+z)=x\cos^2\theta$. On MD lay off $MW=HV$. Erect a perpendicular to MD at W, meeting AB at Q. Erect a perpendicular to AB meeting MD at P.

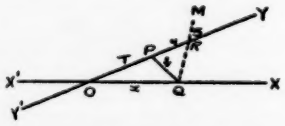
$$MP=MQ\sec\theta=MW\sec^2\theta=x\cos^2\theta\sec^2\theta=x.$$

Hence P is the required point.

No. 228. Proposed by *Walter B. Clarke*, San Jose, California.

Two straight lines XX' and YY' intersect at O so that angle XOY is acute. Q is a variable point on XX' and P a fixed point on OY. The bisector of angle PQX will cut YY' at points outside a certain segment ST of the line. For what angle XOY will $PS=PT$?

Solution by *C. W. Trigg*, Los Angeles City College.



Let $\angle XOY=\theta$, $PO=a$, $PQ=b$, and $OQ=x$. When $\angle PQX=2\theta$, the bisector of $\angle PQX$, QM, is parallel to YY'. As Q approaches O, the intercept of QM and YY' traverses $Y'O$. As Q now recedes along OX', QM is an internal bisector of ΔPQO, in which $PQ>QO$. Hence

the intercept approaches the mid-point, T, of OP as a limit and $PT=\frac{1}{2}a$.

As Q progresses along OX with $\angle PQX>2\Theta$, QM is an external bisector of ΔPQO and meets OY at R. Hence $PR=y=ab/(x-b)$. By the law of cosines, $b^2=a^2+x^2-2ax\cos\Theta$.

$$dy/dx=a^2(x\cos\Theta-a)/b(x-b)^2.$$

Upon equating this derivative to zero, $x=a/\cos\Theta$, for which y is a minimum, i. e., $y=a\sin\Theta/(1-\sin\Theta)=PS$.

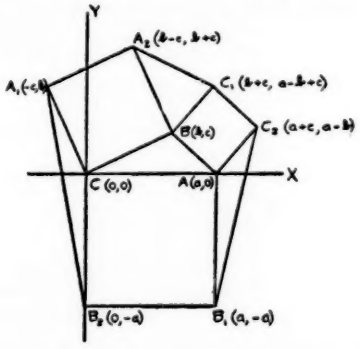
Now if $PS=PT$, $y=\frac{1}{2}a$, so $\sin\Theta=\frac{1}{3}$, and $\Theta=19^\circ\ 28'\ 16''$.

Also solved by the *Proposer*.

No. 244. Proposed by *V. Thébault*, Le Mans, France.

Construct the squares BCA_1A_2, CAB_1B_2, ABC_1C_2 upon the sides of a triangle ABC. Show that the centers of gravity of the areas of the figures made up of the three squares, of the three squares and the triangle ABC, and of the hexagon $A_1A_2C_1C_2B_1B_2$ are collinear.

Solution by *C. W. Trigg*, Los Angeles City College.



Let the vertices of the triangle be $A(a, 0)$, $B(b, c)$ and $C(0, 0)$. Then $AB=\sqrt{(a-b)^2+c^2}=BC_1=\sqrt{(m-b)^2+(n-c)^2}$, where the coordinates of C_1 are (m, n). The slope of AB is $c/(b-a)$, so the slope of BC_1 is $(a-b)/c=(n-c)/(m-b)$. Upon solving these two expressions for m and n, $(m, n)\equiv(b+c, a-b+c)$. Proceeding in like manner, the vertices of the hexagon are identified as

$$C_1(b+c, a-b+c),\ A_2(b-c, b+c),\ A_1(-c,b),\ B_2(0,-a),\ B_1(a,-a),$$

and $$C_2(a+c,a-b).$$

The three bordering triangles each have two sides equal to two sides of the original triangle and the corresponding included angles

are supplementary, so the four triangles have equal areas. The coordinates of the centroid of a triangle are equal to one-third the sum of the coordinates of the vertices of the triangle, and the centroid of a square is the mid-point of its diagonals. Hence,

Polygon	*Centroid*	*Area* (S)
ABC	$(a+b)/3,\ c/3$	$ac/2$
CAB_1B_2	$a/2,\ -a/2$	a^2
AC_2B_1	$(3a+c)/3,\ -b/3$	$ac/2$
ABC_1C_2	$(a+b+c)/2,\ (a-b+c)/2$	$(b-a)^2+c^2$
BA_2C_1	$b,\ (a+3c)/3$	$ac/2$
BCA_1A_2	$(b-c)/2,\ (b+c)/2$	b^2+c^2
CB_2A_1	$-c/3,\ (b-a)/3$	$ac/2.$

The coordinates of the centroid of a group of polygons with areas, S_i, are given by

$$x_c = \sum_{i=1}^{n} x_i S_i \Big/ \sum_{i=1}^{n} S_i, \quad y_c = \sum_{i=1}^{n} y_i S_i \Big/ \sum_{i=1}^{n} S_i,$$

where the centroids of the polygons are (x_i, y_i).

Thus the coordinates of the centroid of the three squares are

$$(x_1, y_1) \equiv \left[\frac{2(a^3+b^3+c^3+bc^2-abc)-a^2b+a^2c+ac^2-ab^2}{4(a^2+b^2+c^2-ab)}, \right.$$
$$\left. \frac{2c(c^2+b^2-ab)+3ab(b-a)+ac(c+a)}{4(a^2+b^2+c^2-ab)} \right].$$

The centroid of the squares and triangle is

$$(x_2, y_2) \equiv \left[\frac{6(a^3+b^3+c^3+bc^2)+3(ac^2-a^2b-ab^2)+4a^2c-5abc}{3[4(a^2+b^2+c^2-ab)+ac]}, \right.$$
$$\left. \frac{6c(c^2+b^2-ab)+9ab(b-a)+ac(4c+3a)}{3[4(a^2+b^2+c^2-ab)+ac]} \right].$$

The centroid of the hexagon is

$$(x_3, y_3) \equiv \left[\frac{6(a^3+b^3+c^3+bc^2)+3a(c^2-b^2-ab)+ac(7a-2b)}{12(a^2+b^2+c^2-ab+ac)}, \right.$$

$$\left. \frac{6c(c^2+b^2-ab)+9ab(b-a)+ac(7c+3a)}{12(a^2+b^2+c^2-ab+ac)} \right].$$

Now $$\frac{y_1-y_2}{x_1-x_2}$$

$$=\frac{2c^3+2b^2c-2abc-9a^2b+9ab^2+3ac^2-a^2c}{2a^3+2b^3+6c^3-3ab^2-ac^2+2bc^2-3a^2b+3a^2c-6abc}$$

$$=\frac{y_1-y_3}{x_1-x_3}.$$

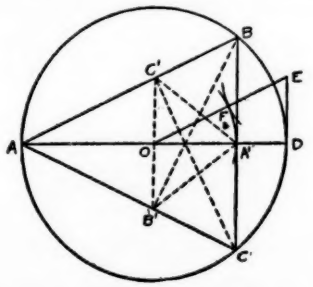
Since the slopes of the two lines joining (x_1, y_1) to the other two centroids are equal, the three centroids are collinear.

No. 247. Proposed by *Walter B. Clarke*, San Jose, California.

Consider only triangles having circumcenter lying on a side of the orthic triangle.

(1) What is the largest possible angle in such a triangle?
(2) Construct one that is isosceles.

Solution by *C. W. Trigg*, Los Angeles City College.



(1) In any obtuse triangle, the orthic triangle and the circumcenter lie on opposite sides of the largest side. Hence the largest possible angle in the given class of triangles is a right angle. In this case the triangle must be right and isosceles, whence the orthic triangle is degenerate and consists of the circumradius to the vertex of the right angle. (The construction of such a triangle is well-known).

(2) Since the circumcenter, O, is on the perpendicular bisector of a side, in an isosceles triangle ABC, it is on the altitude AA' to BC. Hence, except for the case mentioned in (1), O falls on the side $C'B'$ of the

orthic triangle. $C'B'$ is parallel to BC so $AO : (AO+OA') :: C'B' : BC$. Now the ratio of a side of a triangle to the corresponding side of the orthic triangle is equal to the ratio of the circumradius to the distance of the circumcenter from the considered side of the basic triangle*, i. e., $BC : C'B' :: AO : OA'$. Hence, $AO : (AO+OA') :: OA' : AO$. So $OA' = \frac{1}{2}OA(-1+\sqrt{5})$. Such a triangle may be constructed in any circle with radius OA and diameter AD. At D erect $DE \perp AD$ so that $DE = \frac{1}{2}OD$. Draw OE and with ED as radius strike arc cutting OE at F. With FO as the radius and center O strike arc cutting AD at A'. At A' erect perpendicular to AD cutting the circle at B and C. Triangle ABC is the required triangle, for

$$OA' = OF = OE - FE = OE - ED = \sqrt{\overline{ED}^2 + \overline{OD}^2} - ED = \tfrac{1}{2}OD(\sqrt{5}-1).$$

Also solved by *Johannes Mahrenholz.*

No. 248. Proposed by *M. S. Robertson*, Rutgers University.

For $f(x) = a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n$, we have $f(x) = -f(-x)$ if, and only if, a_s vanishes for every even s. What is the corresponding functional equation for which the condition is $a_s = 0$ for $s \equiv 0 \pmod{p}$, $p > 2$?

Solution by the *Proposer.*

It is desired to show that for an analytic function

$$f(x) = \sum_{s=0}^{\infty} a_s x^s, \qquad |x| < R,$$

the necessary and sufficient condition that

(1) $$a_s = 0 \text{ for all } s \equiv 0 \bmod p$$

is that

(2) $$\sum_{k=0}^{p-1} f(xe^{2k\pi i/p}) = 0 \quad \text{identically.}$$

Let the left member of (2) be written

$$\sum_{s=0}^{\infty} a_s x_s \sum_{k=0}^{p-1} e^{2sk\pi i/p}.$$

Now for $s \equiv 0 \bmod p$, $\sum e^{2sk\pi i/p} = p$; while for $s \not\equiv 0 \bmod p$,

$$\sum e^{2sk\pi i/p} = (1 - e^{2s\pi i})/(1 - e^{2s\pi i/p}) = 0.$$

*Altshiller-Court, *College Geometry*, p. 88, (1925).

Thus (2) is reduced to

$$\sum_{j=0}^{\infty} a_{jp}x^{jp}=0 \quad \text{identically.} \tag{2a}$$

In this form it is evident that (2) is true if and only if (1) holds.

PROPOSALS

No. 274. Proposed by *I. C. Nichols*, Louisiana State University.

Show that the flex point of a cubic is its point of symmetry.

No. 275. Proposed by *H. T. R. Aude*, Colgate University.

The sum of two dollars and seventy-one cents is made up of twenty-two coins which are pennies, nickles, dimes, and half-dollars. If there is a different number of each kind is the solution unique?

No. 276. Proposed by *E. P. Starke*, Rutgers University.

Having given an arithmetic progression of real or complex terms with the property that the terms can be rearranged to form a geometric progression, prove that there exists another permutation of the terms which gives a harmonic progression.

No. 277. Proposed by *C. W. Trigg*, Los Angeles City College.

Find a square number of the form $abcdefgh$ with $ab=gh$ and such that $cdef$ is a square number. A solution without the aid of tables is desired.

No. 278. Proposed by *V. Thébault*, Le Mans, France.

In what system of numeration does there exist a square of four digits having the form $aabb=(cd)^2$, with $a+b=c+d$?

No. 279. Proposed by *J. Rosenbaum*, Bloomfield, Connecticut.

If a, b, c, each different from zero, is a solution of

$$px^3+qy^3+rz^3+sxyz=0,$$

where $27pqr+s^3=0$, find a second solution, other than ka, kb, kc, expressed rationally in terms of a, b, c, and the coefficients.

No. 280. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Find the volume of a tetrahedron in terms of its edges.

Bibliography and Reviews

Edited by
P. K. SMITH and H. A. SIMMONS

Living Mathematics. By Ruch, Knight, and Hawkins. Scott, Foresman and Company, Atlanta, 1938. x+576 pages.

The authors do well the work of reviewing arithmetic and introducing the equation up through page 100; at this place is taken a so-called "side trip" into geometry, covering a treatment of congruent triangles *et al*, with some simple problems in construction. These 40 or 45 pages could have come better later in the book, perhaps.

I believe it is a mistake to mark $\angle s$ with small letters, as many authors of junior high school texts do. For instance, they show a triangle like the one in Fig. 1 below when it should be like the one in Fig. 2. If the latter standard were adhered to, it would simplify form very much later on.

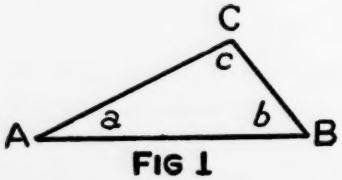


FIG 1

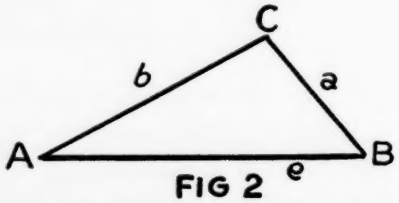


FIG 2

There is unfortunate presentation of material to the effect that savings accounts earn 2% per year; that if one deposits money before the 10th he gets interest from the 1st of that month; and that after the 10th, interest begins at the 1st of the next month. Any 7th or 8th grader will see a mathematical injustice there. Further, unfortunate material is given on "Investing In Stocks." The statement is made that "the people of the United States earn (notice, *earn*, not *make*) billions of dollars each year on their investments in stocks and bonds." It says nothing of the millions of people who lose and are made poor! Poor material to put in a textbook for children!

I am not so sure that it is possible to interest young students, below sophomore age or 10th year, in taxes!

In subtraction, the authors teach to "change the sign," which we have been trying to get away from for years. We, at New Trier High School, say "What added to -3 makes or gives $+10$?" In other

words, "What added to the subtrahend gives the minuend?" Our slogan is "Take the natural view"; then one can almost entirely forget *rules* in early mathematics. One will find the authors' ridiculous directions for subtraction on page 305.

The authors' claim of "integration and socialization" is not well founded, as there is *little* integration and *too much* socialization.

If this book aims to prepare students for "life," the materials for the most part should be offered later in high school. For those in the lowest group and for those not expecting to study higher mathematics and science, the book, with some revision, could be studied profitably in the 12th year. There is little development of material beyond grade school mathematics and this text, which is largely a review, could well come just before the student *plunges into "life."* Our best freshmen might study this book and get something out of it, but they would miss that good and broad foundation for the subsequent study of mathematics and would therefore lose one good year in *their preparation for "life,"* which means college, university, and professional studies, and perhaps some voca ion that means much of science and mathematics. For those not so planning, *Living Mathematics* should be given in the 12th year. Freshmen studying it will lose it before they enter "life", and furthermore most of them have had adequate amount of that kind of mathematics in the grammar school and need a change. They are as saturated as they can be at that age and need two or three more years of maturity before studying it again. It will be more difficult to teach this material to freshmen than to teach them straight algebra.

Some simple forms of algebra and geometry could be taught in the early years while the student is becoming mature enough to appreciate this work, which is essentially a small amount of glorified commercial arithmetic.

There is sufficient material in the book to keep the average poor student busy for three semesters. I do not see how it could be covered in one year. We increase trouble for ourselves in trying to teach too much to this slower group. If we could ever get administrators to require five or six years for the slower group to graduate, instead of pushing them through with the upper group in four years, we would really accomplish something.

In algebra and geometry, we set up a standard and if the student does not come close to it, he is marked as failing, and must repeat the course.

In work like *Living Mathematics*, practically no standard is recognized, because the work does not serve as a prerequisite for further

study, and consequently all students are passed even though they accomplish very little. Then we say "That is what we should teach because no one fails or very few fail." But should we?

It seems to me that we should teach high school students *how* to think, reason, and analyze, and not so much *what* to think and to memorize.

New Trier High School. W. A. SNYDER.

Plane Geometry. By George T. Major. Charles Scribner's Sons, New York, 1938. xv + 280 pages.

This book consists of thirty lessons, an appendix of additional theorems, difficult examples for special study, a set of tables, and an index.

The first five lessons call the pupils attention to his need of a new vocabulary in the study of geometry, make more exact the meaning of the terms he already uses, and show that exact knowledge is more reliable than assumption. In these lessons a few geometric constructions such as bisecting an angle and drawing one line perpendicular to another are included along with a sample of formal proof.

In lessons 6, 7, 8 the student proves a few theorems by doing what he is told to do, giving reasons for being able to do so, and drawing conclusions from what he has done. Lessons 10, 11, 12 take up theorems on congruent triangles, parallel lines, sum of the angles of polygons, and problems applying these principles in their solution. Lesson 13 is devoted to quadrilaterals. The locus and its applications are introduced in lesson 15. Lessons 9, 14, 23, 30 are review lessons. Parallel lines cutting off proportional segments are discussed in Lesson 16. Lessons 17, 18, 19 take up similarity. Lessons 20, 21 are devoted to trigonometry. The study of trigonometry includes the complete solution of any triangle. Lesson 22 brings in the geometry of circles. In lessons 24, 27, 28 areas of polygons, circles, and portions of circles are studied. In these lessons use is made of trigonometry and logarithms. Lesson 29 consists of constructions reviewing theorems on circles.

In the first of the book the numbering of the theorems is slightly confusing. In Lessons 4 and 5 three theorems on congruency are stated. One is proved in Lesson 10 and numbered Theorem 9, but neither of the other two are numbered; yet one is proved, and both are used extensively. In Lesson 11 it is stated that the approach to the student has had three theorems on congruency.

This book is a departure from the traditional geometry text book. In the study of many of the theorems the student is led by construction to arrive at the fact of the theorems and to prove the theorem before it is stated. The arrangement of the book is novel, as is the approach to the study of its material.

The reviewer believes that the student wou'd enjoy the study of this book, that it would stimulate his interest in geometry, and that he would subsequently not find geometry the formidable subject that it is sometimes pictured.

Louisiana Polytechnic Institute. H. F. SCHRŒDER.

Coordinate Solid Geometry. By Robert J. T. Bell. Macmillan, London, 1938. lvi + 175 pages.

This book contains the first nine chapters of Professor Bell's *Elementary Treatise on Coordinate Geometry of Three Dimensions.* It is designed "for students whose interest in pure mathematics will not extend beyond the requirements of pass degree examinations in mathematics." The theory of lines, planes, and quadrics is developed in classical manner. Oblique coordinates are freely used. Separate chapters are devoted to plane sections of quadrics and to rulings of quadrics. In addition to the wealth of problems throughout the text, two sets of miscellaneous exercises contain a total of 186 problems. Right-handed axes are used consistently.

On page 23, e ample 10, the statement is made that for small angles the sine of the angle equals the angle. In article 33 the unnecessary restriction is made that p be positive in the equation of a plane.

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p.$$

Since p is merely the algebraic measure of the projection of a vector on an oriented line, desirable generality is lost without any compensating advantage. The use of homogeneous coordinates, the distinction between metric and projective properties of figures, and vector analysis (covariant and contravariant vectors in connection with oblique axes) would have enriched the discussions without altering the length of the exposition. However, the book does give an ample source of problem material.

Virginia Military Institute. W. E. BYRNE.

Introductory Mathematical Analysis. By Joel S. Georges and Jacob M. Kinney. The Macmillan Company, New York, 1938. xv+594 pages. $3.00.

This text has for its central idea the concept of a function. It is a unified course in which the concepts of algebra, trigonometry, analytic geometry, and the calculus are used as tools in studying the elementary algebraic and transcendental functions. The first three chapters include linear functions and equations in one, two, and three variables. In these chapters the analytic geometry of the straight line and plane is developed. The algebra of linear equations is woven into this analytic geometry. This procedure gives the algebra a needed concreteness. The delta notation of the calculus is introduced with the definitions of the slope of a straight line and rate of change. In Chapter IV the power function, ax^n, is discussed with its relation to variation, logarithms, and the parabola. The next two chapters treat the quadratic function and the polynomial of the nth degree, each in one unknown. The elements of differentiation and integration are treated. Chapter VII on "Circular Functions" covers the analytic geometry of the circle and introduces the student to trigonometry as circular functions. Exponential functions, the theory of annuities, and the differentiation and elementary integration of the exponential functions are found in Chapter VIII. The algebra and the analytic geometry of the quadratic function in two variables is given in Chapter IX. The calculus is used in discussing tangents to the conics. Analytic trigonometry, complex numbers, polar coordinates, and parametric representation are the topics, mainly, in the next three chapters. The principal part of the text closes with Chapter XIII, which contains thirty pages on solid analytic geometry. Fifty-three pages of tables are included.

This text has very splendid features. It has both logical and pedagogical appeal. The authors state that the text might be covered in two semesters of five hours weekly. An exceptional class might do this. There is serious doubt as regards the average class—especially if the students are required to take the course and come with poor preparation.

The illustrative examples and theorems are numbered serially beginning with the firs of the text. Throughout the text appear sets of cumulative reviews. This seems to be a splendid treatise. It should be examined by anyone desiring a unified text.

This text would be more useful, if a set of answers had been given. On page 37, Illustrative Example 10, had the authors answered the question asked, the answer would have been 1.137π instead of 36.38π.

On page 150, Illustrative Example 60, the use of the t and x are confused; the differentiation should have been with respect to t or the given equation written $y=16x^2$. On page 323, Illustrative Example 153, the apparent accuracy of the answer \$537.63 is not justified, due to the fact that a four-place annuity table was used.

The format of the text is especially pleasing, and the figures throughout are excellent.

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THE undersigned has a limited number of autograph signed letters of Felix Klein, which he has decided to sell. These documents are in excellent condition, and suitable for framing in a lecture room or study. More information will be supplied upon request. Those interested should apply to

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